Supplemental to A Constraint-based Formulation of Stable Neo-Hookean Materials

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CCS CONCEPTS

• **Computing methodologies → Physical simulation**.

KEYWORDS

finite element method, physically-based animation, elasticity, real-time physics

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1 CONSTRAINT GRADIENTS

We now give the gradients of our constraint functions to ease implementation. In general our constraint functions are defined in terms of the deformation gradient $\mathbf{F}$, not the particle positions $\mathbf{x}$. To make the connection between the two, we discretize the equations using FEM in conjunction with constant strain tetrahedra. Let $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4$ be the initial positions of the particles adjacent to a tetrahedron and $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ their current positions. With the matrices

$$\mathbf{X} = [\mathbf{x}_1 - \mathbf{x}_4, \mathbf{x}_2 - \mathbf{x}_4, \mathbf{x}_3 - \mathbf{x}_4] \quad \text{and} \tag{1}$$

$$\bar{\mathbf{X}} = [\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_4, \bar{\mathbf{x}}_2 - \bar{\mathbf{x}}_4, \bar{\mathbf{x}}_3 - \bar{\mathbf{x}}_4] \tag{2}$$

we can express the uniform deformation gradient within the tetrahedron as

$$\mathbf{F}(\mathbf{x}) = \mathbf{X}\bar{\mathbf{X}}^{-1}. \tag{3}$$

For the volume conservation constraint

$$C_H(\mathbf{x}) = \det(\mathbf{F}) - 1 \tag{4}$$

we get

$$[\nabla_{\mathbf{x}_1}, \nabla_{\mathbf{x}_2}, \nabla_{\mathbf{x}_3}] C_H(\mathbf{x}) = [\mathbf{f}_2 \times \mathbf{f}_3, \mathbf{f}_3 \times \mathbf{f}_1, \mathbf{f}_1 \times \mathbf{f}_2]\, \mathbf{Q}^T, \tag{5}$$

where the $\mathbf{f}_i$ are the columns of $\mathbf{F}$ as before and $\mathbf{Q}$ the rest state matrix. For the deviatoric constraint

$$C_{\mathrm{D}}(\mathbf{F}) = \sqrt{|\mathbf{f}_1|^2 + |\mathbf{f}_2|^2 + |\mathbf{f}_3|^2} \tag{6}$$

the gradients are

$$[\nabla_{\mathbf{x}_1}, \nabla_{\mathbf{x}_2}, \nabla_{\mathbf{x}_3}] C_{\mathrm{D}}(\mathbf{x}) = \frac{1}{r_S}\, [\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3]\, \mathbf{Q}^T, \tag{7}$$

where $r_S = \sqrt{|\mathbf{f}_1|^2 + |\mathbf{f}_2|^2 + |\mathbf{f}_3|^2}$. As with the stress tensor, the gradients have a singularity at fully collapsed state which is prevented by the volume conservation constraint.

In all cases the gradient with respect to $\mathbf{x}_4$ is the negative sum of the ones for $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$.

2 NEO-HOOKEAN THEOREM

The deviatoric energy term of the Neo-Hookean model looks quite arbitrary at first, however, we now prove the statement

$$\forall \mathbf{A} \in \mathbb{R}^{3\times3} : \det(\mathbf{A}) = 1 \wedge \mathrm{tr}(\mathbf{A}^T\mathbf{A}) = 3 \leftrightarrow \mathbf{A} \text{ is a rotation matrix}, \tag{8}$$

which we call the Neo-Hookean theorem. It shows that the deviatoric term (second condition) is precisely what needs to be added to a volume conservation constraint (first condition) to drive the model to its non-inverted rest state modulo translation and rotation without the need of a polar decomposition.

We show that

$$\boxed{\forall \mathbf{A} \in \mathbb{R}^{3\times3} : \det(\mathbf{A}) = 1 \wedge ||\mathbf{A}||_F^2 = 3 \iff \mathbf{A} \text{ is a rotation matrix}}, \tag{9}$$

where $||.||_F$ is the Frobenius norm. We first prove the implication from left to right. Let $\mathbf{A} = \mathbf{U\Sigma V}$ be the singular value decomposition of $\mathbf{A}$, where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and Σ a diagonal matrix with non-negative entries λ_i. Since $||\mathbf{A}||_F = ||\mathbf{MA}||_F = ||\mathbf{AM}||_F$ for orthogonal matrices $\mathbf{M}$ we get

$$3 = ||\mathbf{A}||_F^2 = ||\mathbf{U\Sigma V}||_F^2 = ||\Sigma||_F^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2. \tag{10}$$

We have

$$1 = \det(\mathbf{A})^2 = \det(\Sigma)^2 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{11}$$

Now by the inequality of arithmetic and geometric means

$$1 = \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{3} \geq \sqrt[3]{\lambda_1^2\lambda_2^2\lambda_3^2} = 1. \tag{12}$$

This inequality is only an equality if $\lambda_1 = \lambda_2 = \lambda_3$. Therefore $\lambda_i = 1$ and $\Sigma = \mathbf{I}$. It follows that $\mathbf{A} = \mathbf{UV}$ and therefore, orthogonal. Since $\det(\mathbf{A}) = +1$, $\mathbf{A}$ is a rotation matrix as well. □

For the implication from right to left: A rotation matrix has $\det(A) = 1$. Also, if $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ are the column vectors of $\mathbf{A}$ then $||\mathbf{A}||_F^2 = |\mathbf{a}_1|^2 + |\mathbf{a}_2|^2 + |\mathbf{a}_3|^2 = 3$. □

3 COMPUTATION OF THE STRESS TENSORS

The stress induced by Ψ_{D} is simply,

$$\sigma_{\mathrm{Spherical}} = \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} = \mu \mathbf{F}. \tag{13}$$

The stress induced by the hydrostatic term has the form $p\mathbf{I}$, where p is the scalar pressure and $\mathbf{I}$ the identity matrix. Since we handle the hydrostatic term as a hard constraint, we have to derive the pressure from the Lagrange multiplier λ. In [Macklin et al. 2016] the

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authors show that the force acting along on a distance constraint can be derived from λ as $\mathbf{f} = \lambda / h^2$. In our case, we can derive the hydrostatic stress via

$$\sigma_H = \frac{\lambda_H}{h^2 V_{\text{tet}}} \mathbf{I}, \tag{14}$$

where V_{tet} is the volume of the tetrahedron.

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