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Primal/Dual Descent Methods for Dynamics

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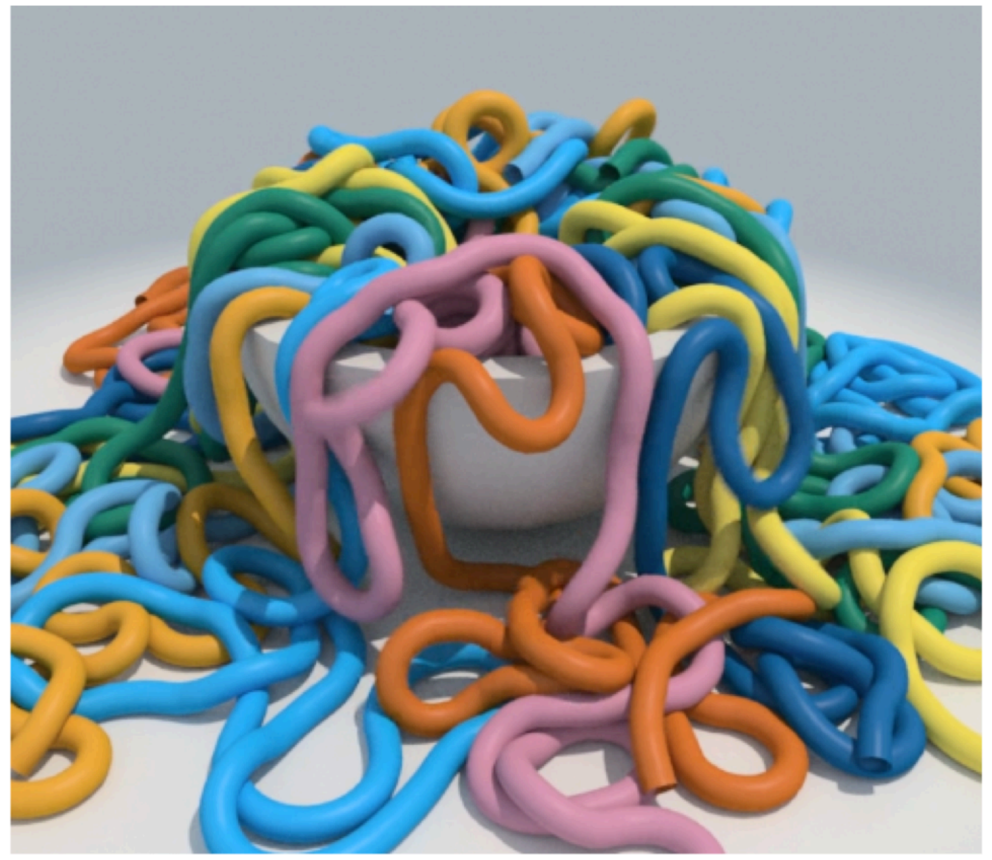
MOTIVATION

- Two common views of implicit time integration:
  - Projective Dynamics [Bouaziz et al. 14] (primal)
  - Position-based Dynamics [Müller et al. 07] (dual)
- Questions:
  - How are they related?
  - How sensitive are they to ill-conditioning?
  - Can we extend Projective Dynamics to large-scale rigid body simulation?



VARIATIONAL IMPLICIT EULER

- Primal optimization problem

$$g(\mathbf{u}) \equiv \frac{1}{2}\left(\mathbf{u}-\tilde{\mathbf{u}}\right)^T \mathbf{M}\left(\mathbf{u}-\tilde{\mathbf{u}}\right) + \sum_i U_i\left(\mathbf{q}^+(\mathbf{u})\right)$$

- Time-stepping Update:

$$\mathbf{u}^+ = \boxed{\underset{\mathbf{u}}{\operatorname{argmin}}\, g(\mathbf{u})}$$

$$\mathbf{q}^+ = \mathbf{q}^- + \Delta t \mathbf{G}\mathbf{u}^+$$

u = Velocity

q = Position

G = Kinematic map

PRIMAL DESCENT METHODS

- Fixed-point iteration:

$$\mathbf{u}_{i+1}^{+} \leftarrow \mathbf{u}_{i}^{+} - \alpha \mathbf{P} \frac{\partial g}{\partial \mathbf{u}}^{T}$$

- Can choose preconditioner P, to approximate Hessian inverse:
  - Quasi Newton [Liu et al. 17]
  - Gauss Newton [Bouasziz et al. 14]
  - Jacobi [Wang et al.16]
  - Constant [Desbrun et al. 99]

- Leads to different forms of Projective Dynamics

Newton:

$$\mathbf{P}^{N} = \left(\frac{\partial^2 g}{\partial \mathbf{u}^2} \right)^{-1}$$

Gauss-Newton:

$$\mathbf{P}^{GN} \equiv \left[\mathbf{M} + \Delta t^2 \, k \, \mathbf{J}^T \mathbf{J} \right]^{-1}$$

Jacobi:

$$\mathbf{P}^{D}_{dd} \equiv \frac{1}{\mathbf{M}_{dd} + \Delta t^2 \, k \, \mathbf{J}^2_d}$$

Constant:

$$\mathbf{P}^{S}_{dd} \equiv \frac{1}{\mathbf{M}_{dd} + \Delta t^2 \sum_i k_i}$$

DUAL PROBLEM

- Can reformulate as a constrained optimization problem by factorizing potentials into parts

$$U = -\frac{1}{2}c(\mathbf{q})\lambda$$

- Introduce Lagrange multipliers
- Lagrange dual function:

$$h(\boldsymbol{\lambda}) = \inf_{\mathbf{u}} \mathcal{L}(\mathbf{u}, \boldsymbol{\lambda}) = \mathcal{L}(\mathbf{u}^*, \boldsymbol{\lambda})$$

Primal Problem:

$$\mathbf{u}^+ = \operatorname*{argmin}_{\mathbf{u}} g(\mathbf{u})$$

Dual Problem:

$$\boldsymbol{\lambda}^+ = \operatorname*{argmax}_{\boldsymbol{\lambda}} h(\boldsymbol{\lambda})$$

DUAL ASCENT

- Fix multipliers, solve primal optimization
- Dual Ascent step on the Lagrange multipliers

- Approximate the primal minimization
- Use a Jacobi preconditioner for ascent step
- Leads to XPBD formulation

$$\mathbf{P}^D_{ii} = \frac{1}{\Delta t^2 \mathbf{J}_i \mathbf{M}^{-1} \mathbf{J}_i^T + \mathbf{K}_{ii}^{-1}}.$$

Dual Ascent:

$$\mathbf{u}_{k+1} = \underset{\mathbf{u}}{\operatorname{argmin}}\, \mathcal{L}(\mathbf{u}, \boldsymbol{\lambda}_k)$$

$$\boldsymbol{\lambda}_{k+1} = \boldsymbol{\lambda}_k + \alpha \frac{\partial h}{\partial \boldsymbol{\lambda}}^T$$

XPBD:

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \Delta t \mathbf{M}^{-1} \mathbf{J}^T \boldsymbol{\lambda}_k$$

$$\boldsymbol{\lambda}_{k+1} = \boldsymbol{\lambda}_k + \alpha \mathbf{P}^D \frac{\partial h}{\partial \boldsymbol{\lambda}}^T$$

CONDITIONING

Primal Problem:

$$\mathbf{u}^+ = \operatorname*{argmin}_{\mathbf{u}} g(\mathbf{u})$$

Primal Hessian:

$$\frac{\partial^2 g}{\partial \mathbf{u}^2} = \left[\cancel{\mathbf{M}} + \Delta t^2 \mathbf{J}^T \mathbf{K} \mathbf{J}\right]$$

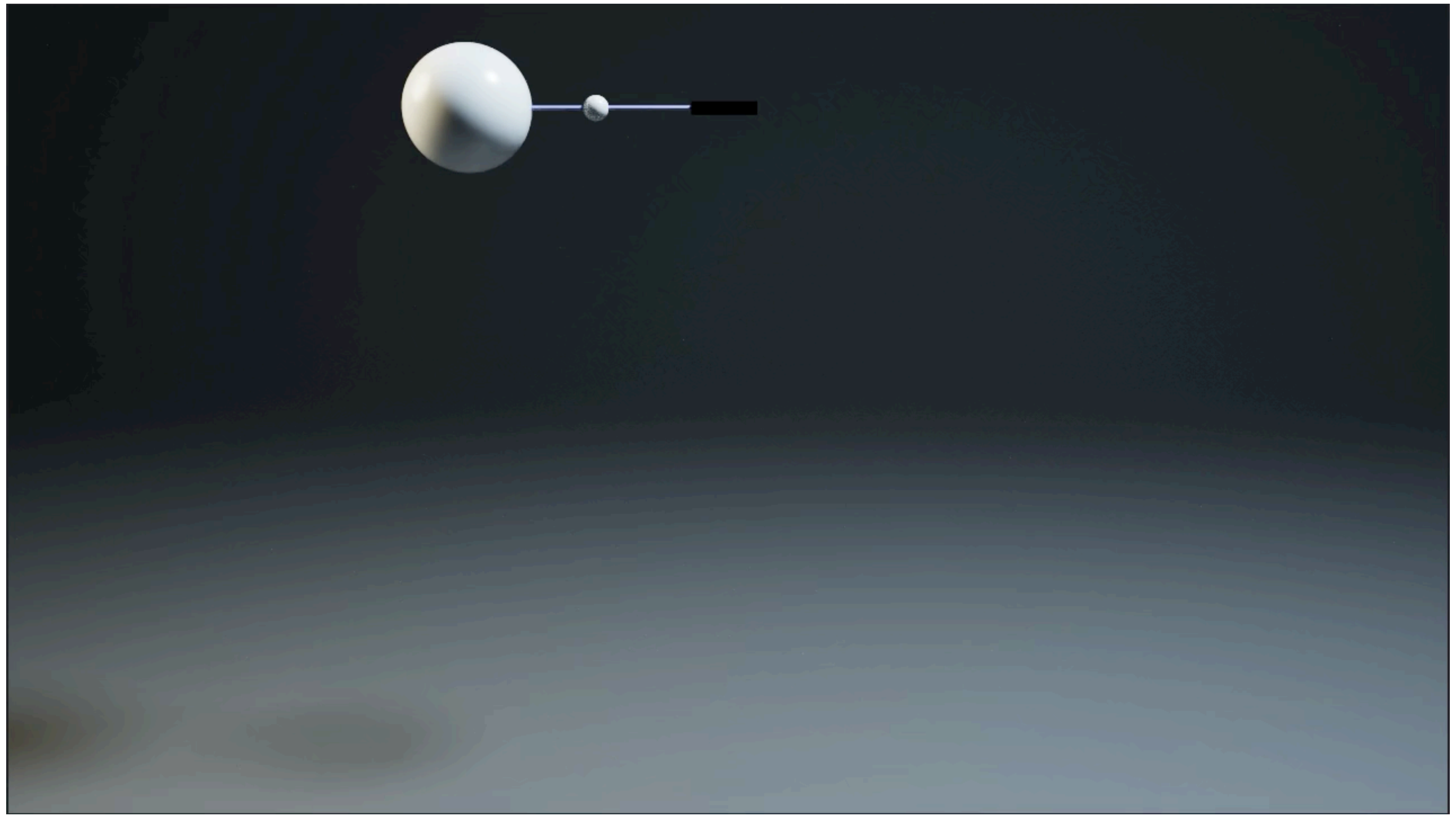
Dual Problem:

$$\boldsymbol{\lambda}^+ = \operatorname*{argmax}_{\boldsymbol{\lambda}} h(\boldsymbol{\lambda})$$

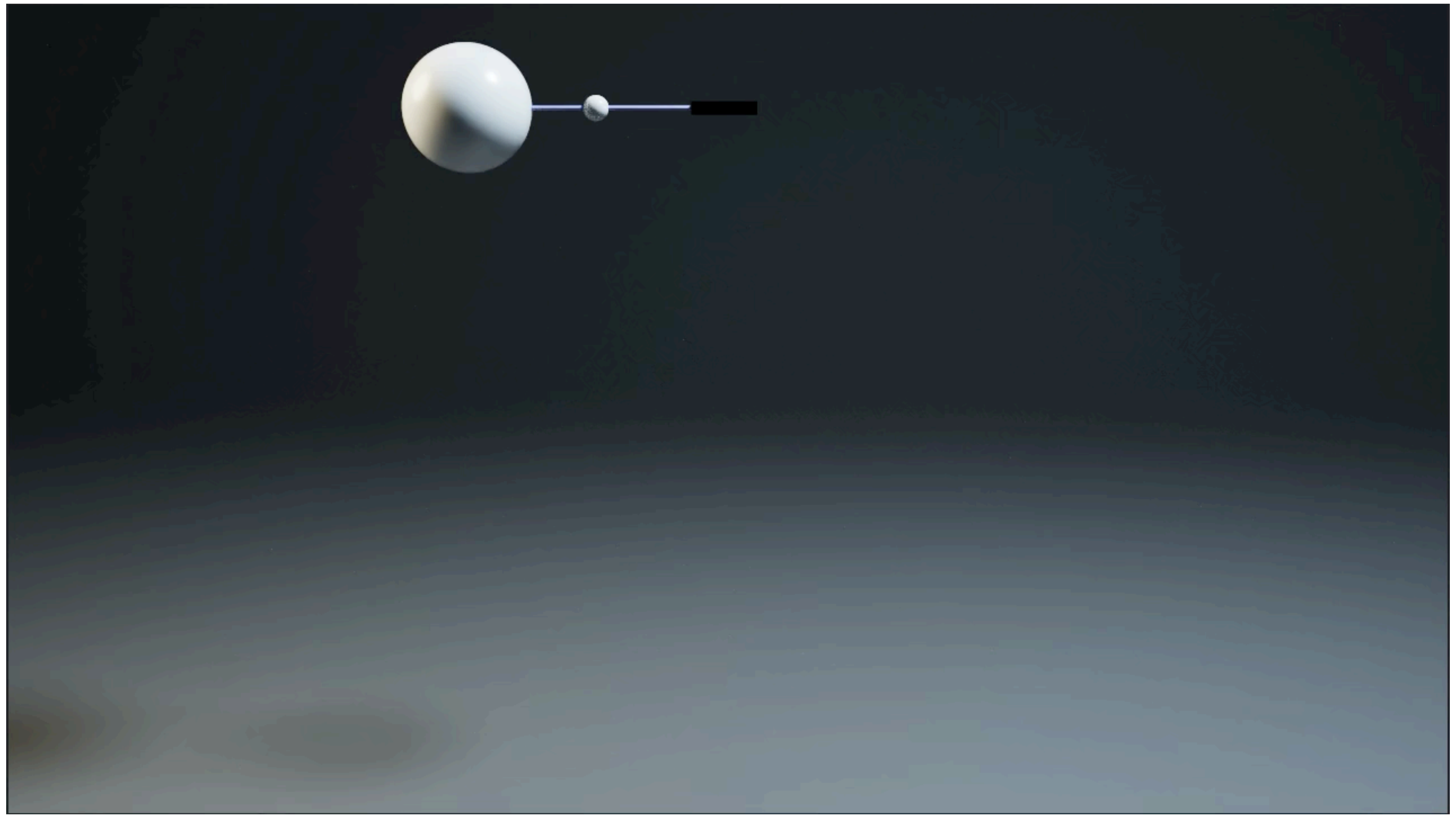
Dual Hessian:

$$\frac{\partial^2 h}{\partial \boldsymbol{\lambda}^2} = -\left[\Delta t^2 \mathbf{J} \mathbf{M}^{-1} \mathbf{J}^T + \cancel{\mathbf{K}^{-1}}\right]$$

MASS RATIO SENSITIVITY



Primal



Dual

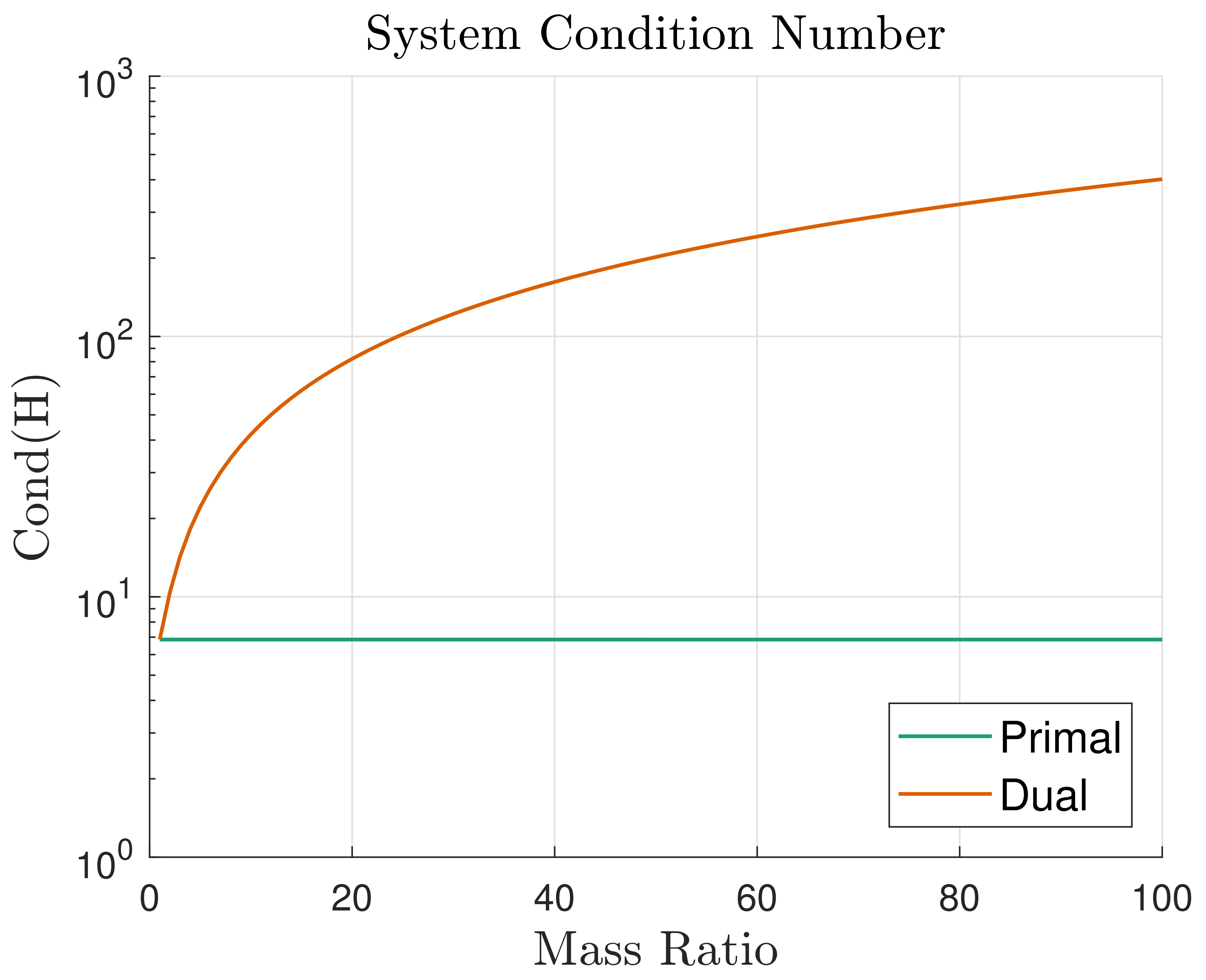
STIFFNESS RATIO SENSITIVITY

Primal

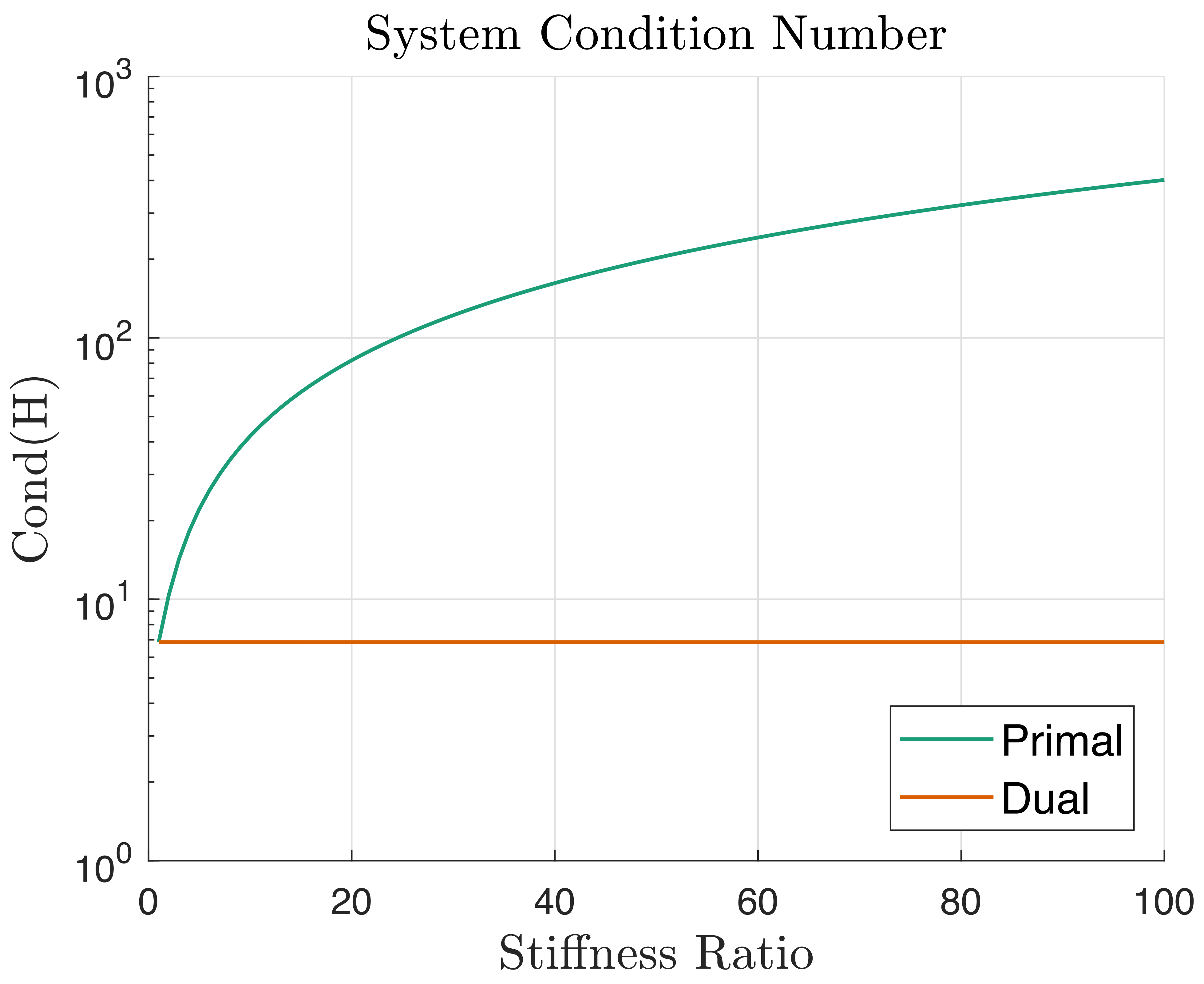
Dual

CONDITIONING

Mass Ratio



Stiffness Ratio



CONTACT

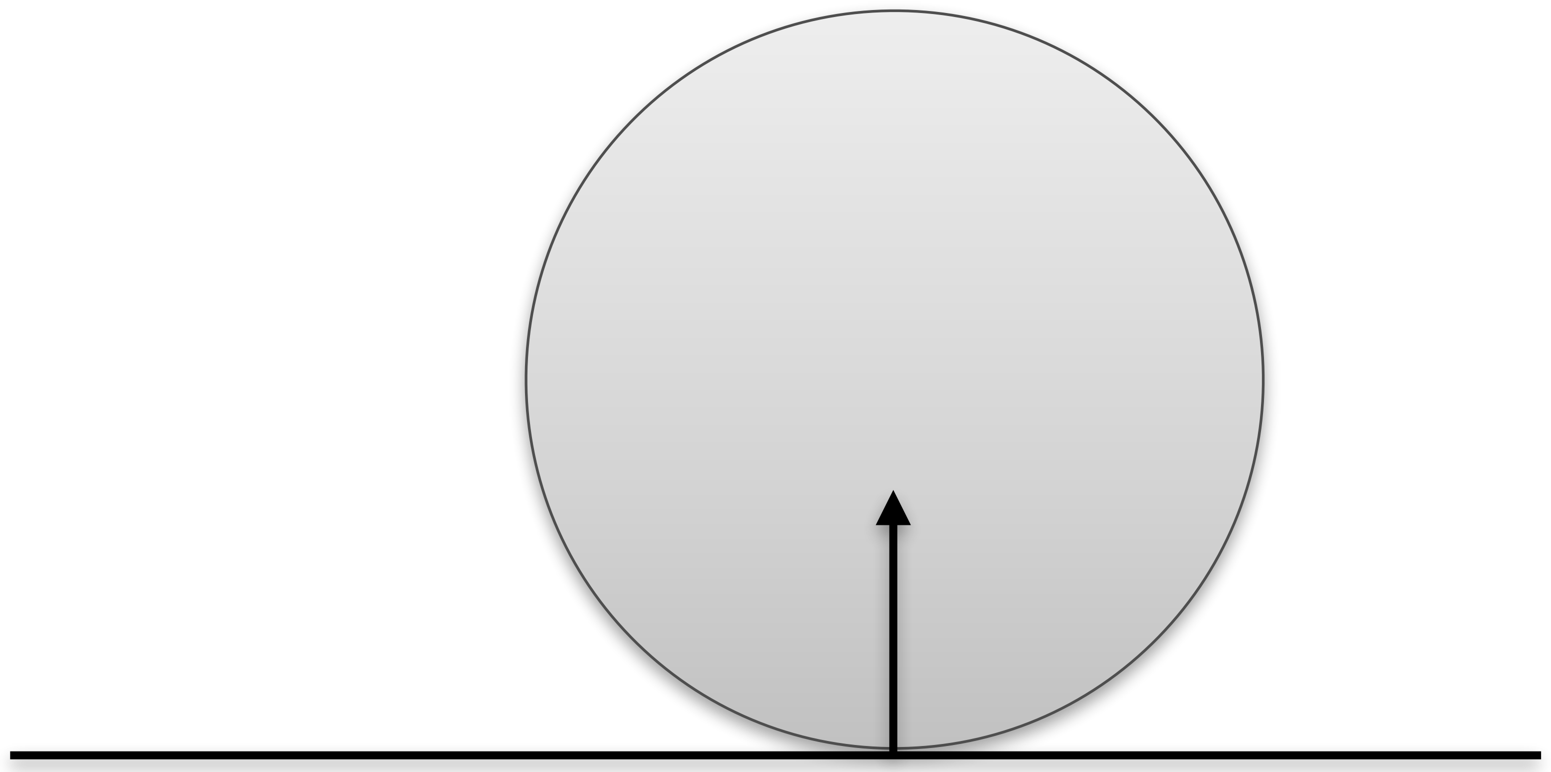
DUAL CONTACT

- Hard contact constraint, zero interpenetration
- Complementarity problem:

$$C_n(\mathbf{q}) = \mathbf{n}^T \left[\mathbf{a}(\mathbf{q}) - \mathbf{b}(\mathbf{q})\right] - d$$

$$0 \leq \lambda \perp C_n \geq 0$$

- Enforce as a bound constraint on lambda > 0 on ascent step

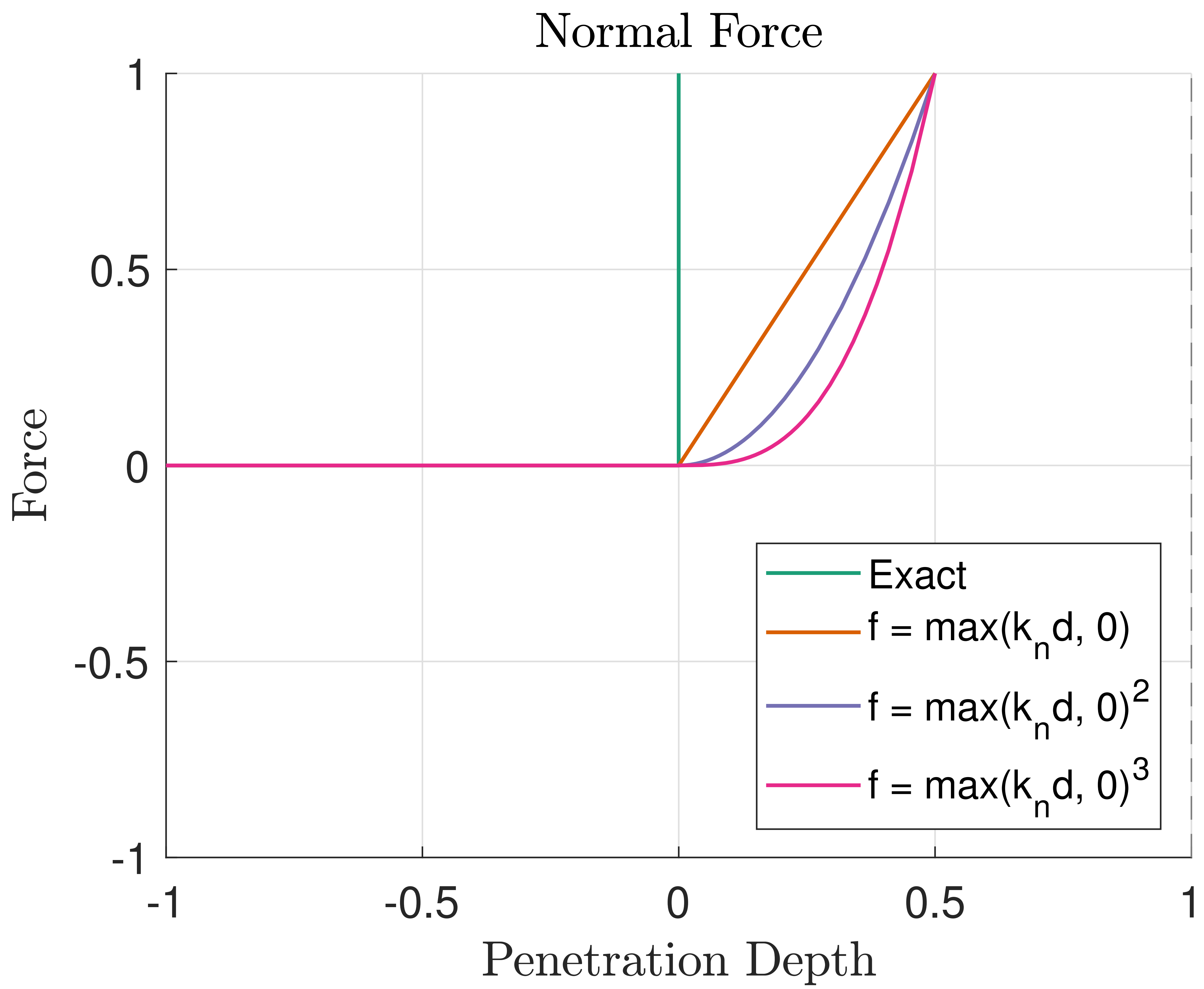


PRIMAL CONTACT

- Simple relaxed model:

$$U_n(\mathbf{q}) \equiv \frac{k_n}{p}\min(0, C_n(\mathbf{q}))^p$$

- Exponentiate to obtain smoothness
- Analytic derivatives
- Many force models, e.g.: Hertzian:
- Large class of implicit penalty methods

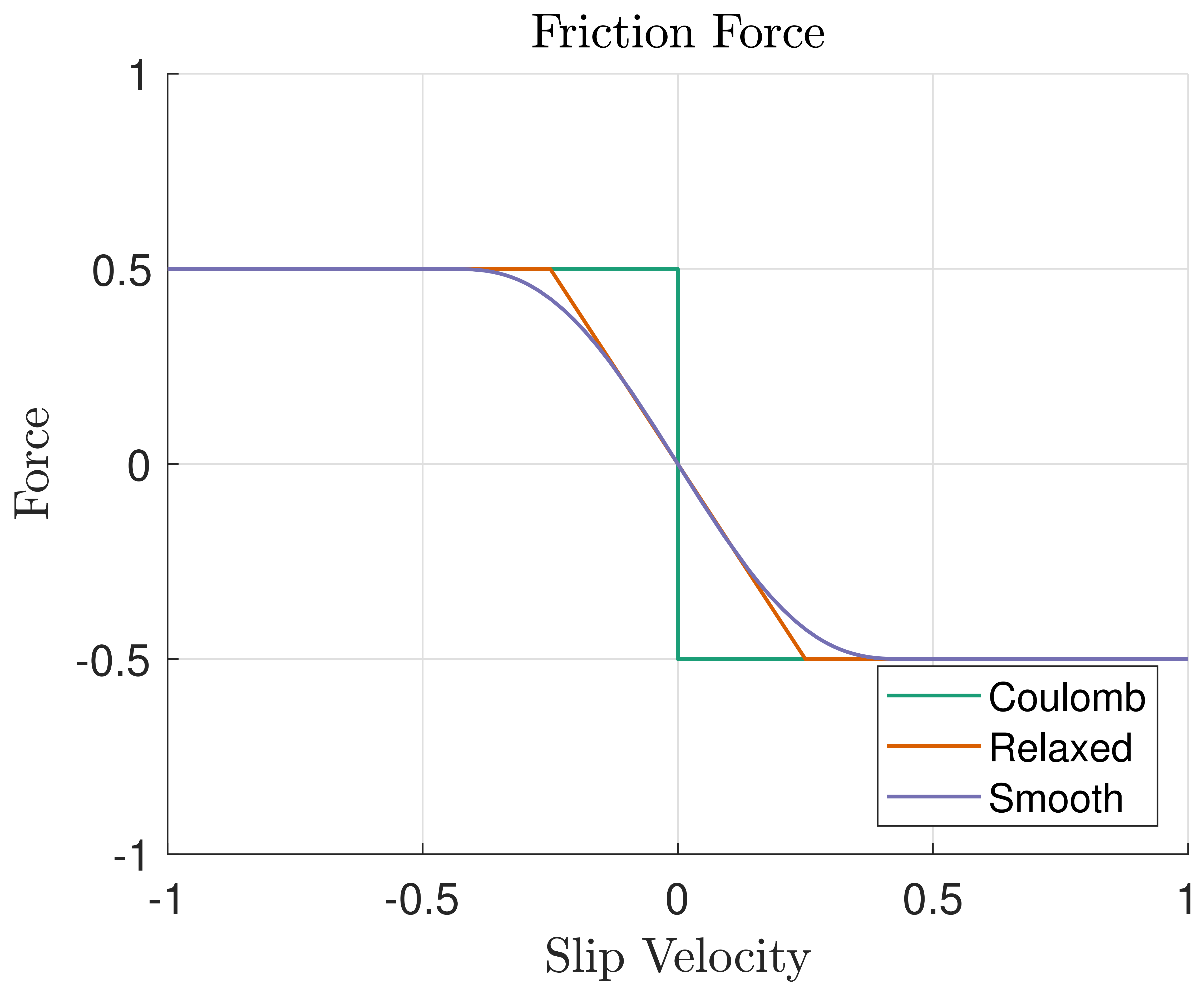


PRIMAL FRICTION

- Simple relaxed model:

$$\mathbf{f}_f(\mathbf{u}) \equiv -\min\left(k_f, \mu\frac{|\mathbf{f}_n|}{|\mathbf{u}_s|}\right)\mathbf{D}^T\mathbf{u}_s,$$

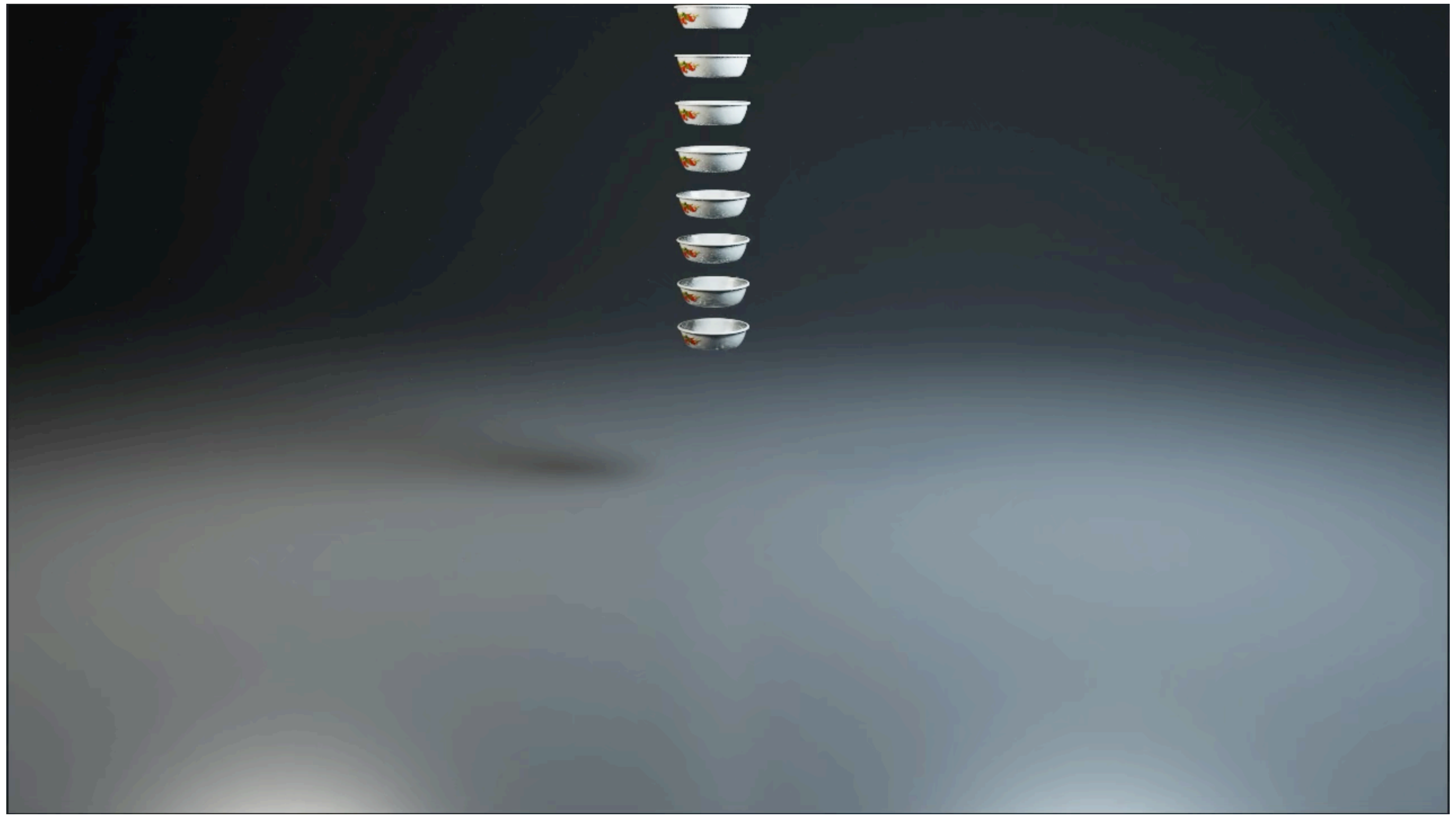
- Always allows some slipping, how much depends on k_f
- Can be derived from a non-smooth potential (in paper)
- Can also use smooth versions, e.g.: pseudo-Huber, tanh, etc



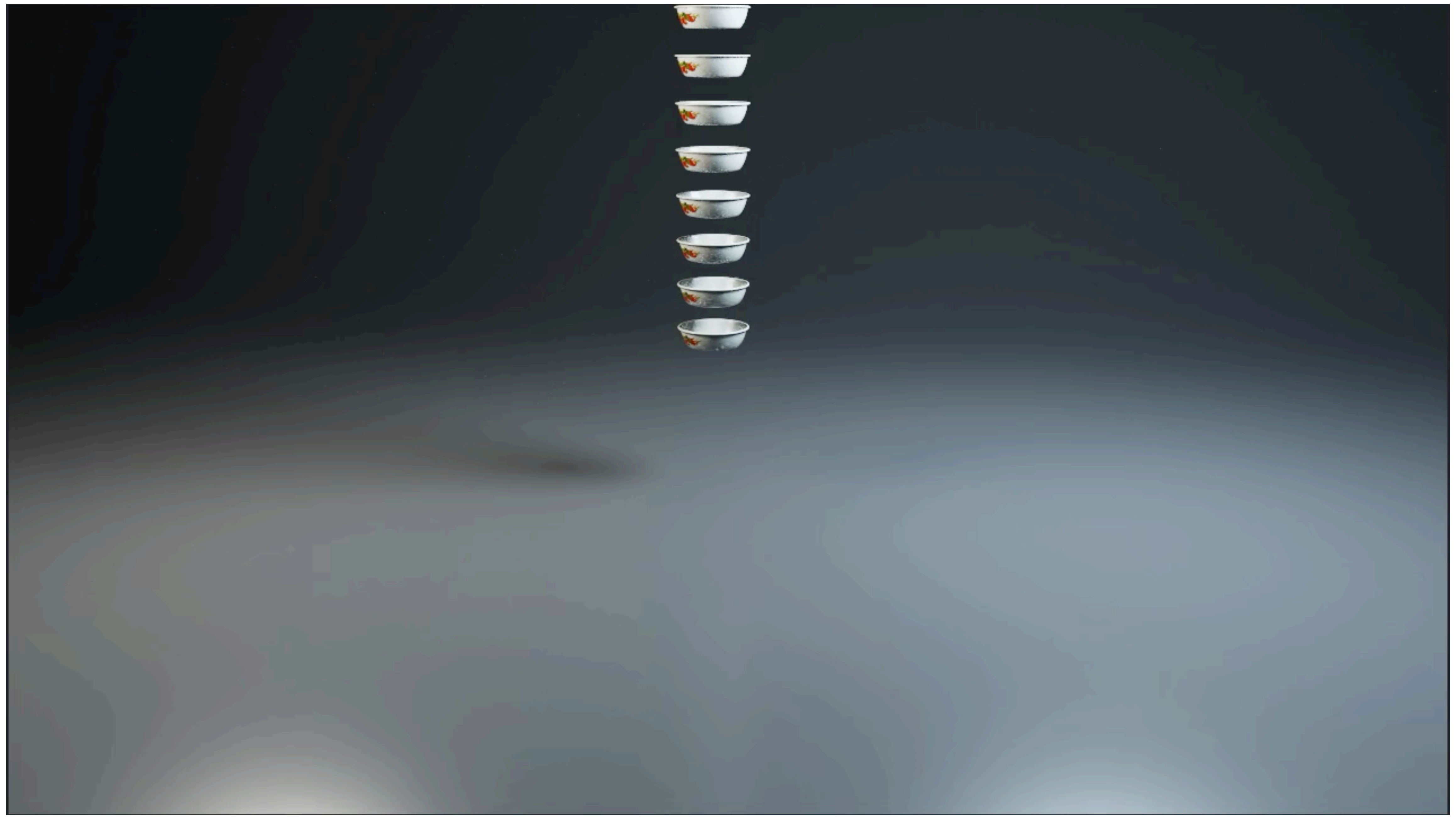
QUESTIONS

- Can penalty methods of contact achieve similar accuracy?
- Stability for large time-steps?
- How does sensitivity manifest in contact?
- Force distributions, differentiability?

WELL-CONDITIONED



Primal



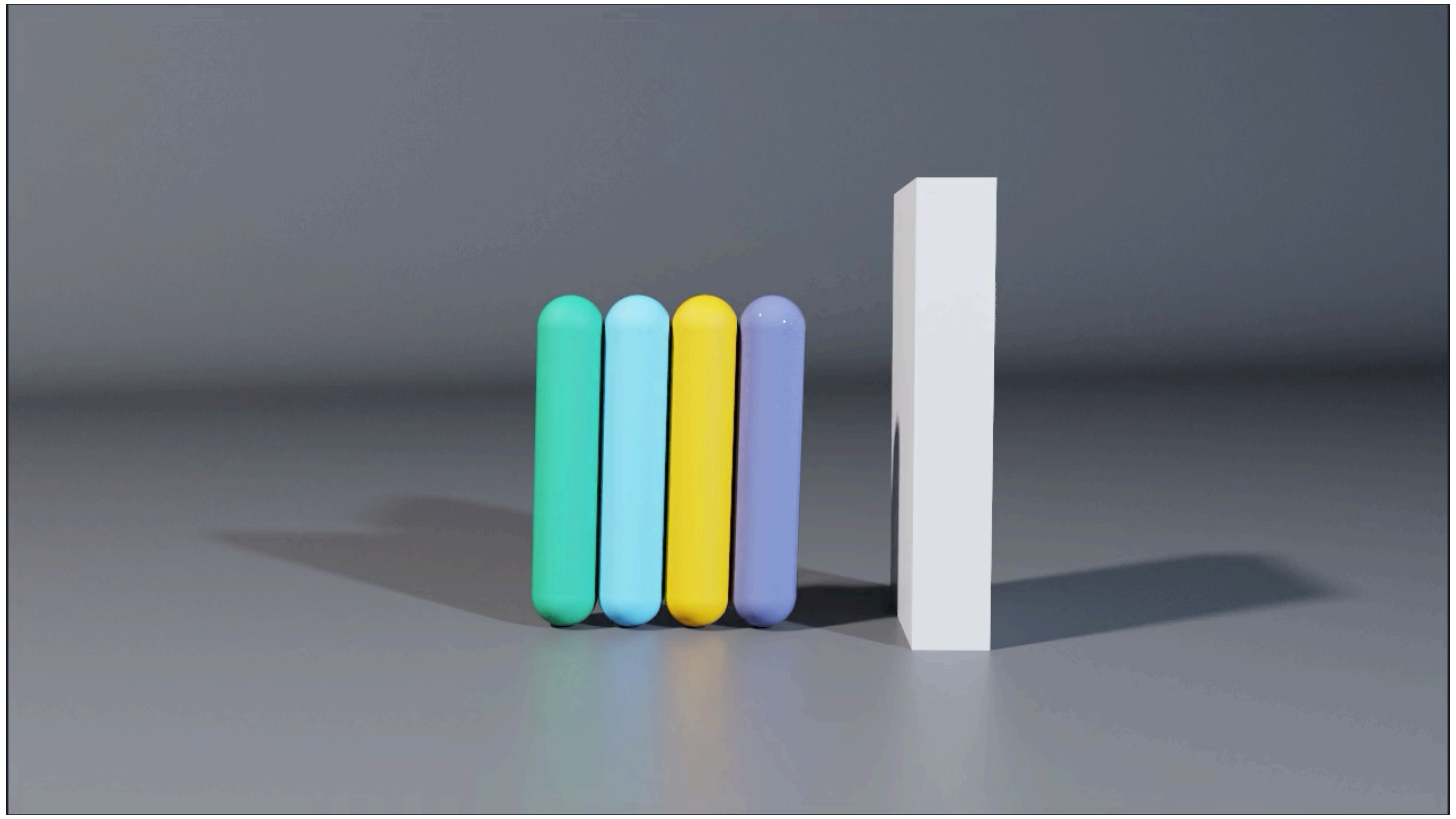
Dual

ILL-CONDITIONED (MASS RATIO)

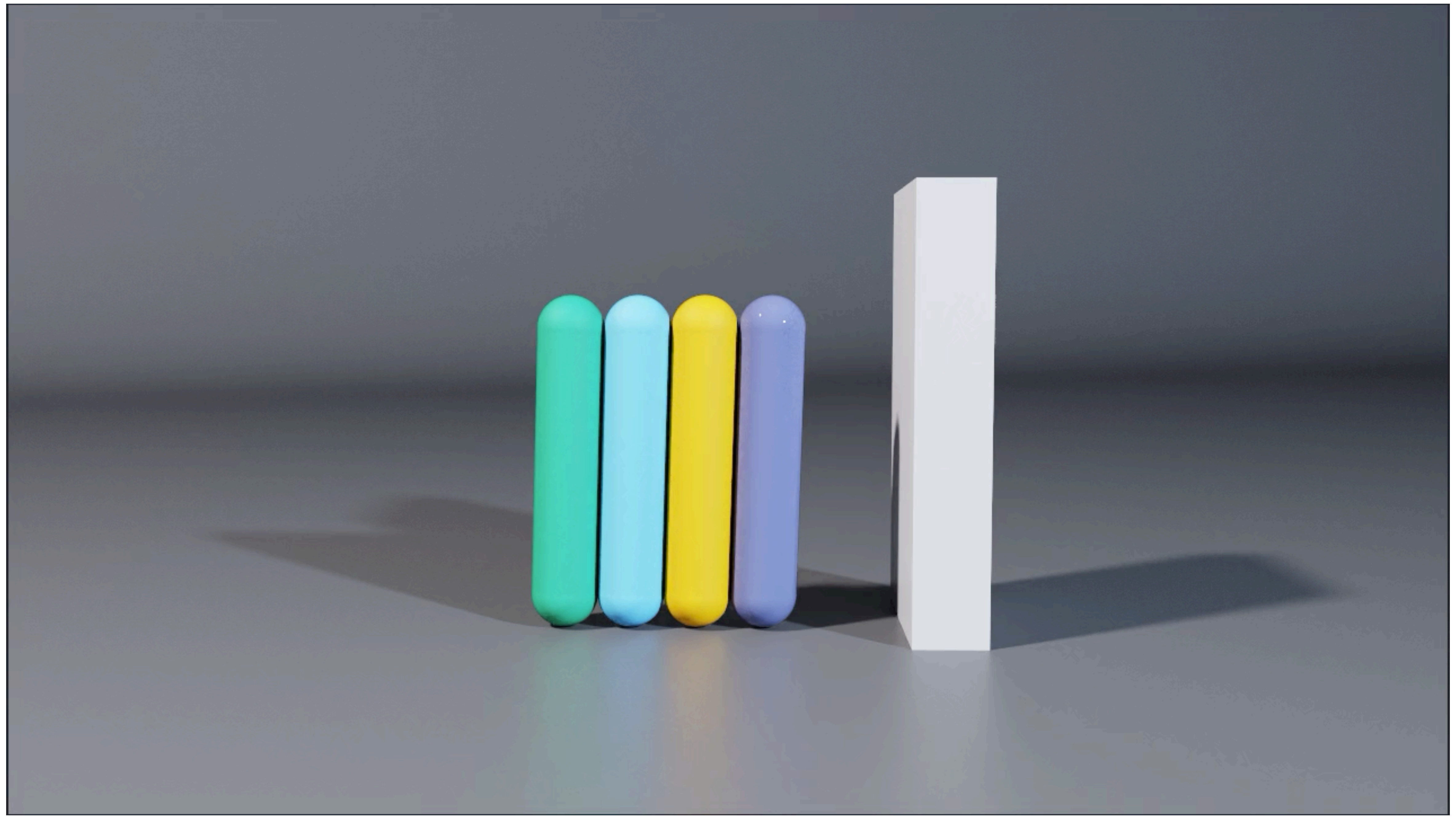
Primal

Dual

ILL-CONDITIONED (STIFFNESS RATIO)

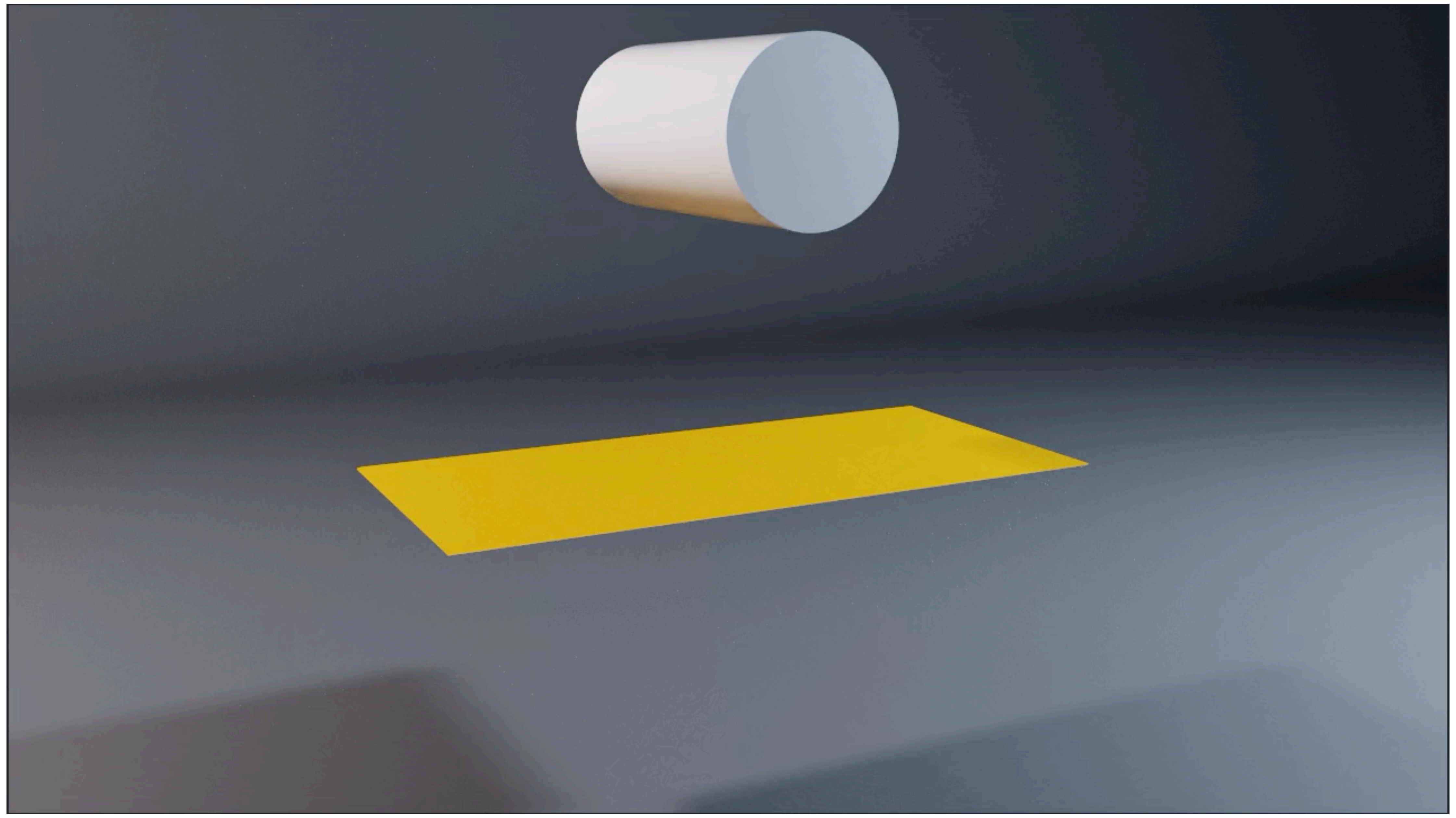


Primal

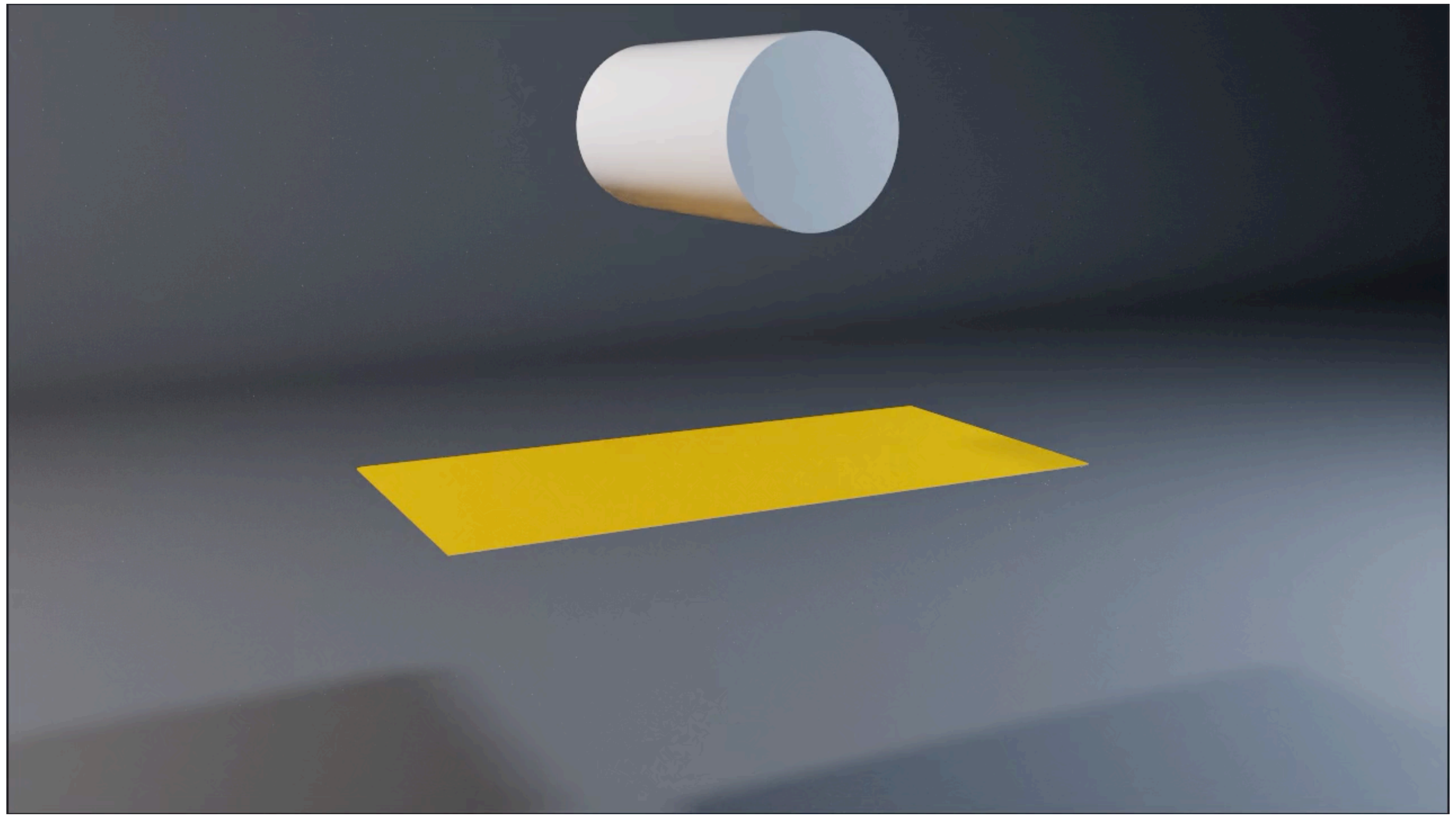


Dual

CLOTH



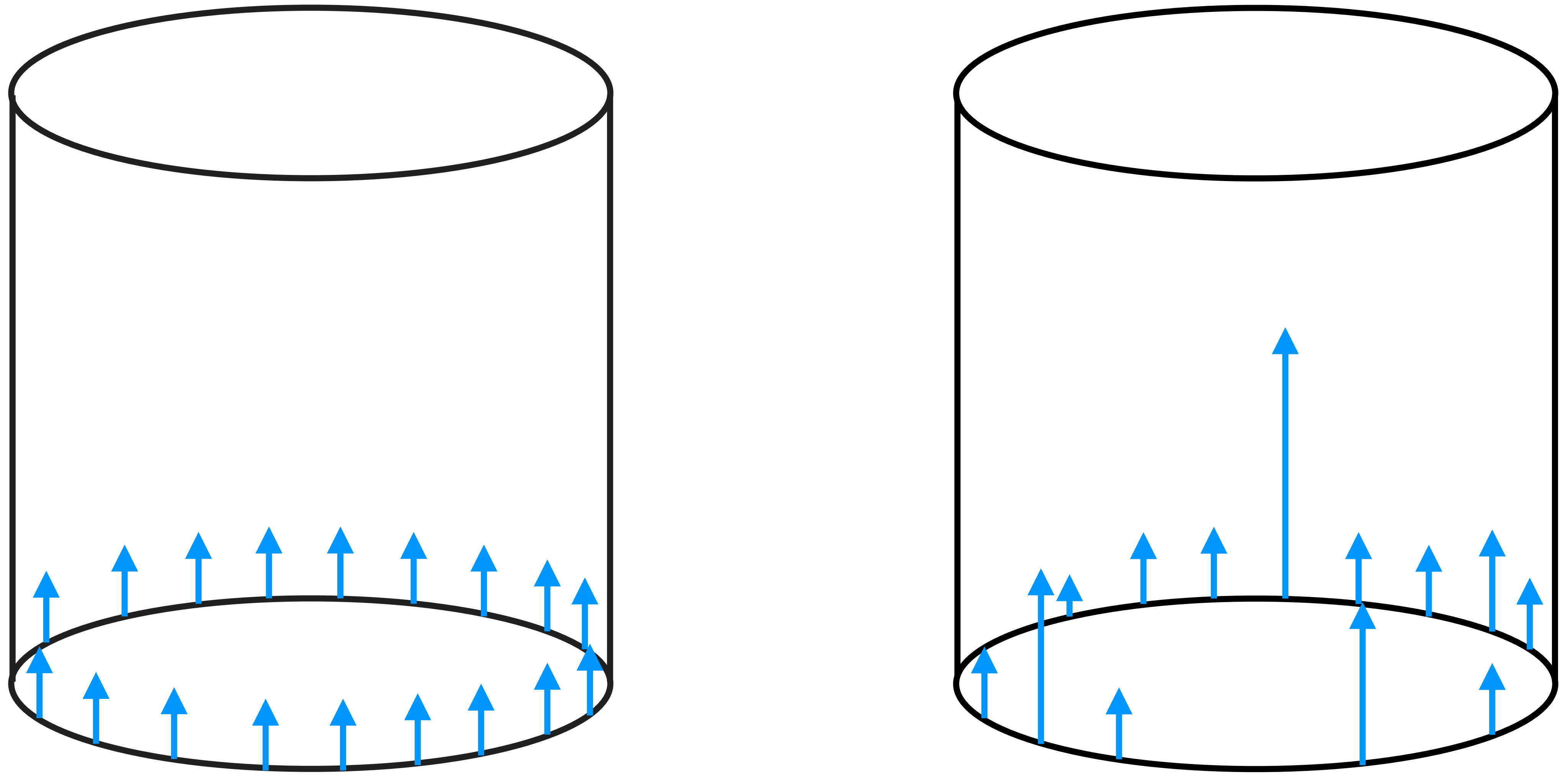
Primal



Dual

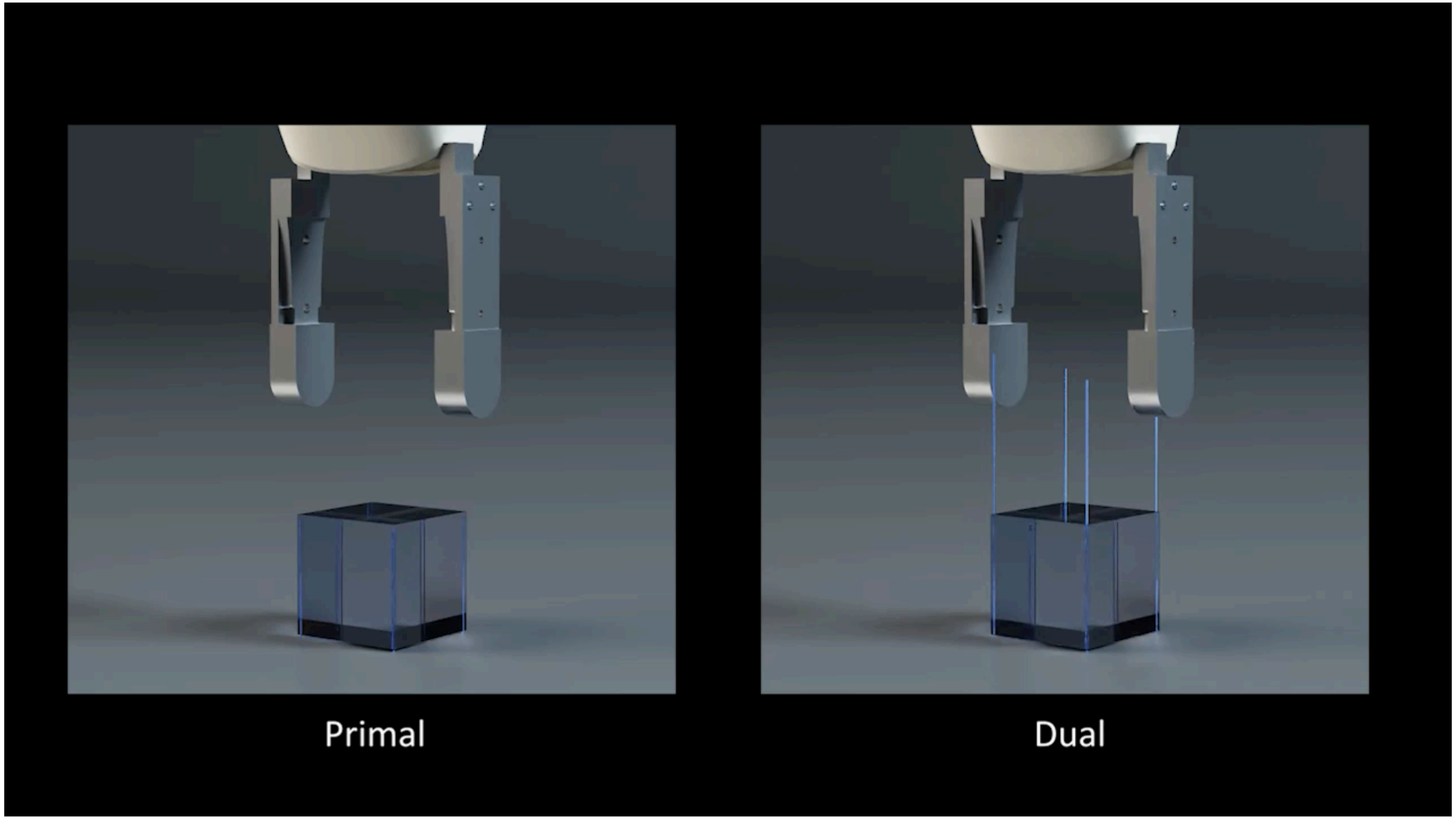
UNDERDETERMINED PROBLEMS

- When J is rank deficient problem is ill-posed, may be:
- Underdetermined and consistent
- > 1 valid solution
- e.g.: linearly dependent contacts
- Leads to noisy contact force distribution for Gauss-Seidel type solvers

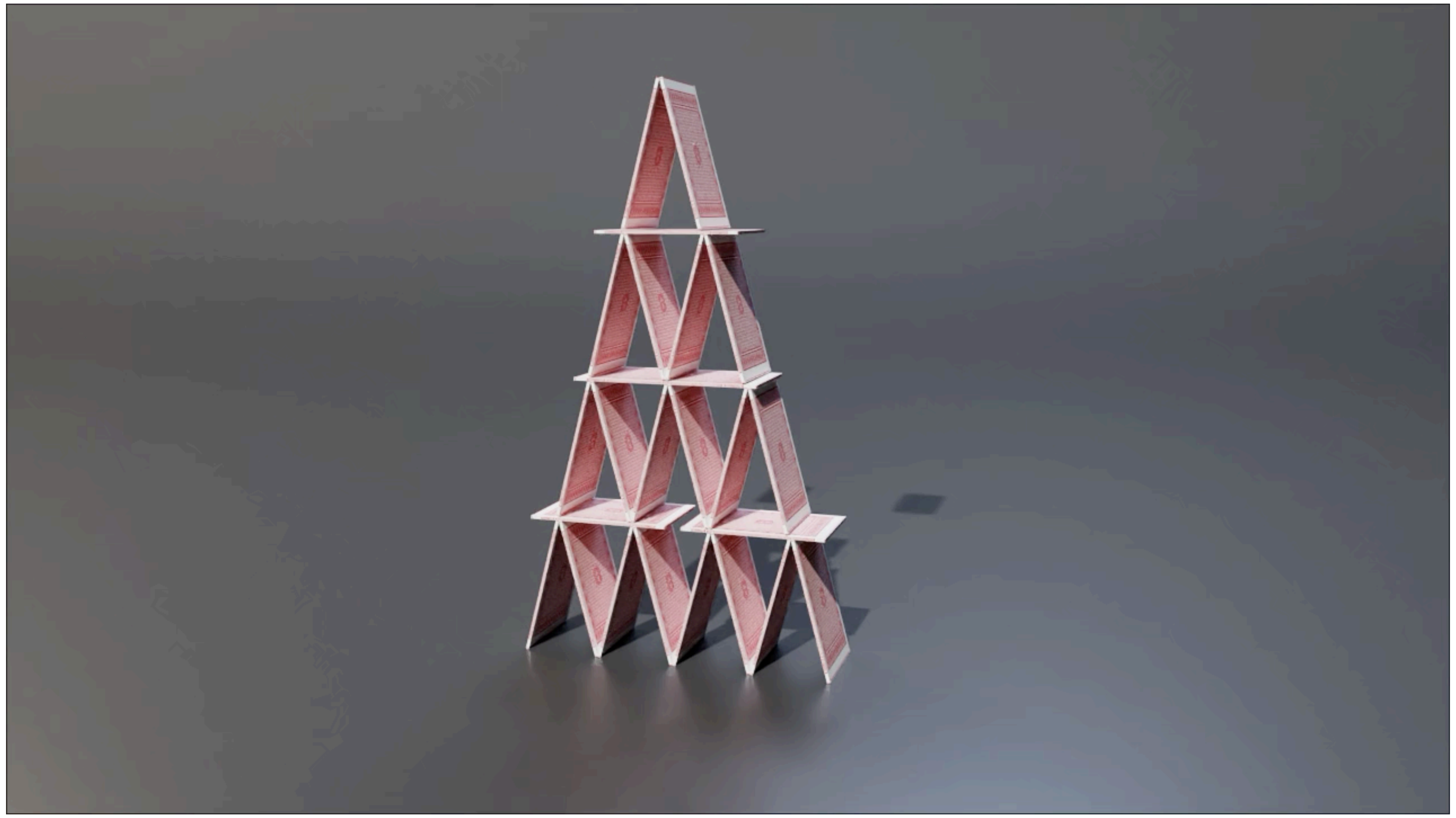


Two valid solutions

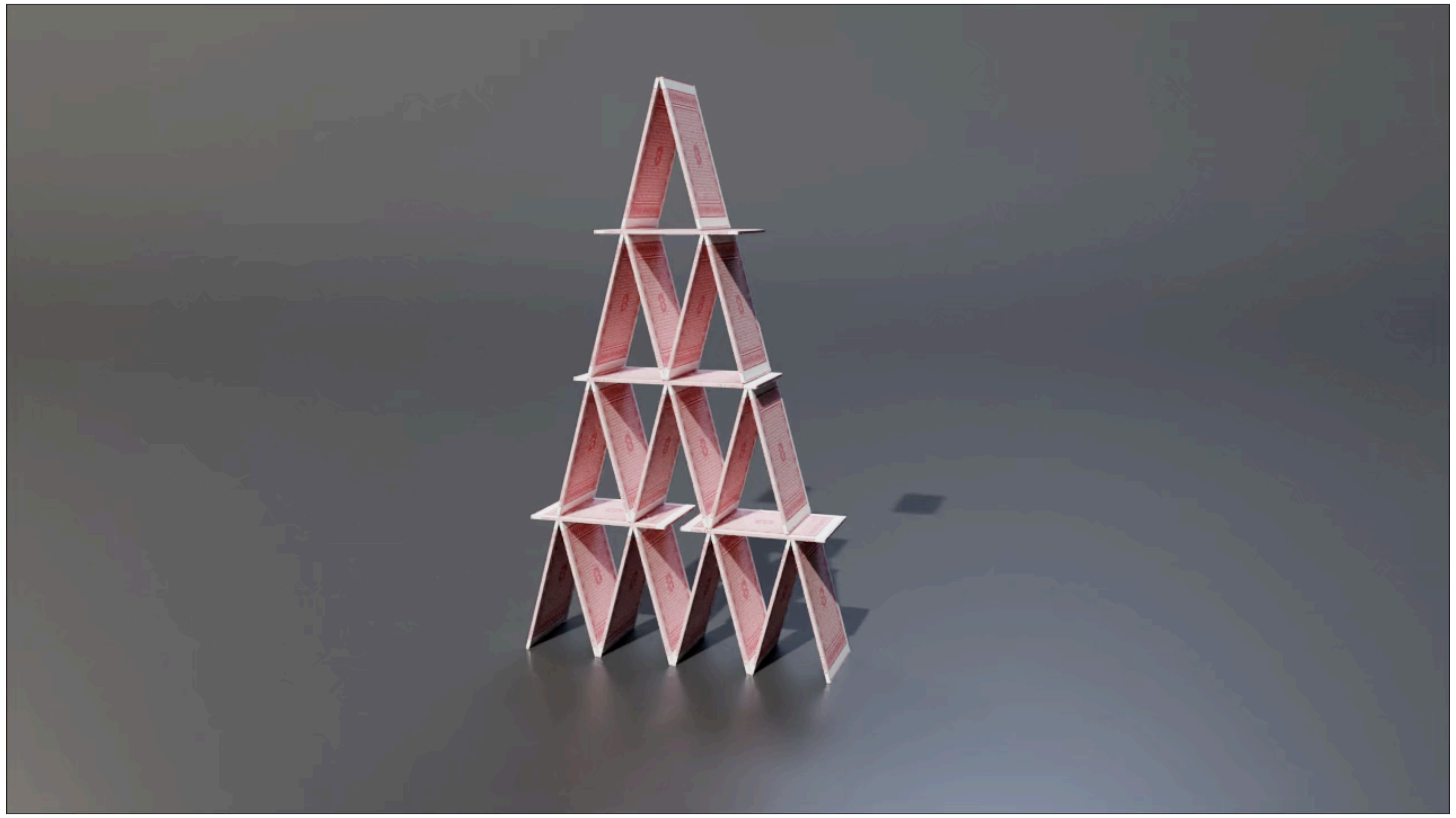
FORCE DISTRIBUTIONS



STABILITY - CARDS

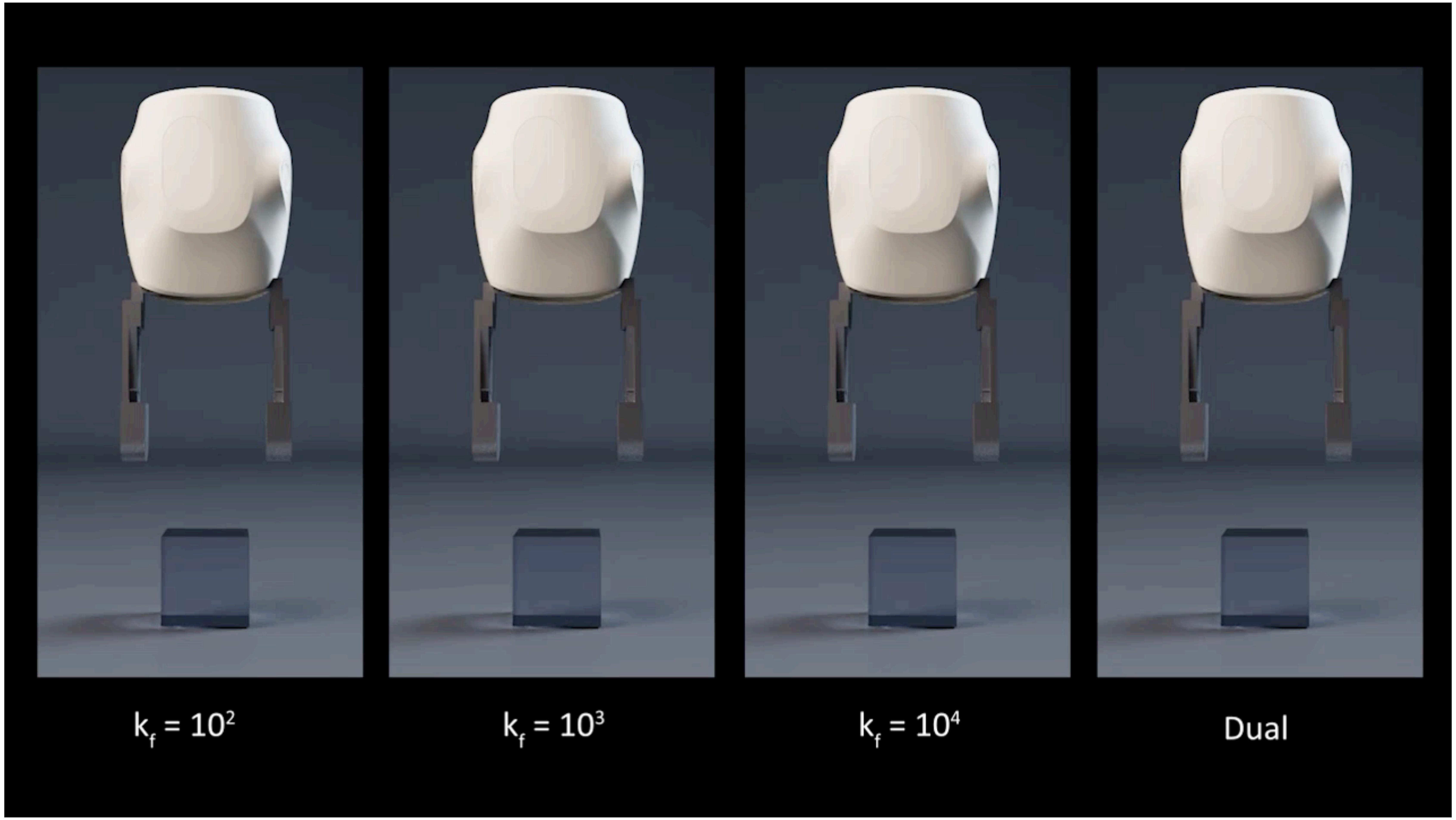


Primal

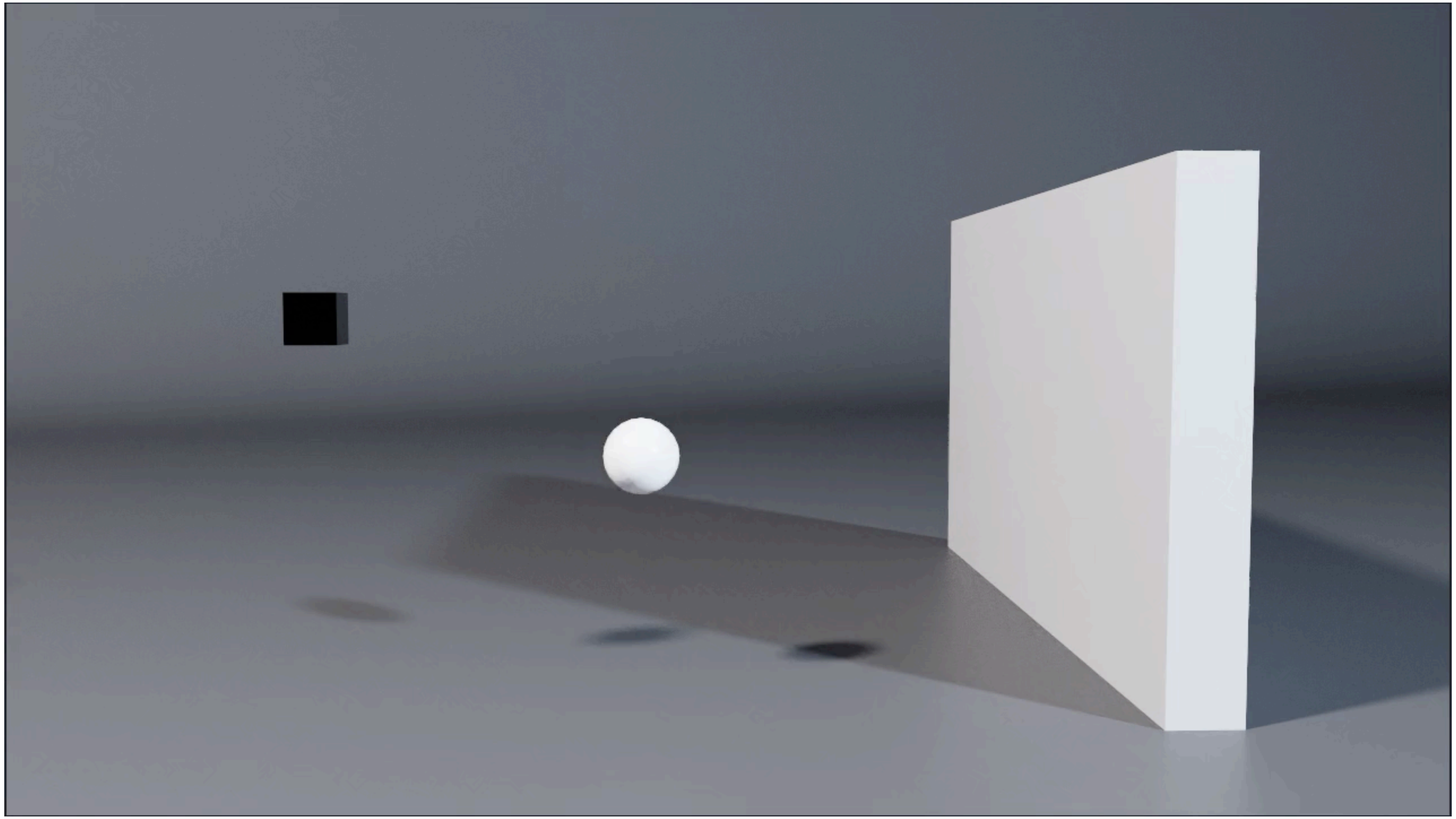


Dual

STABILITY - GRASPING

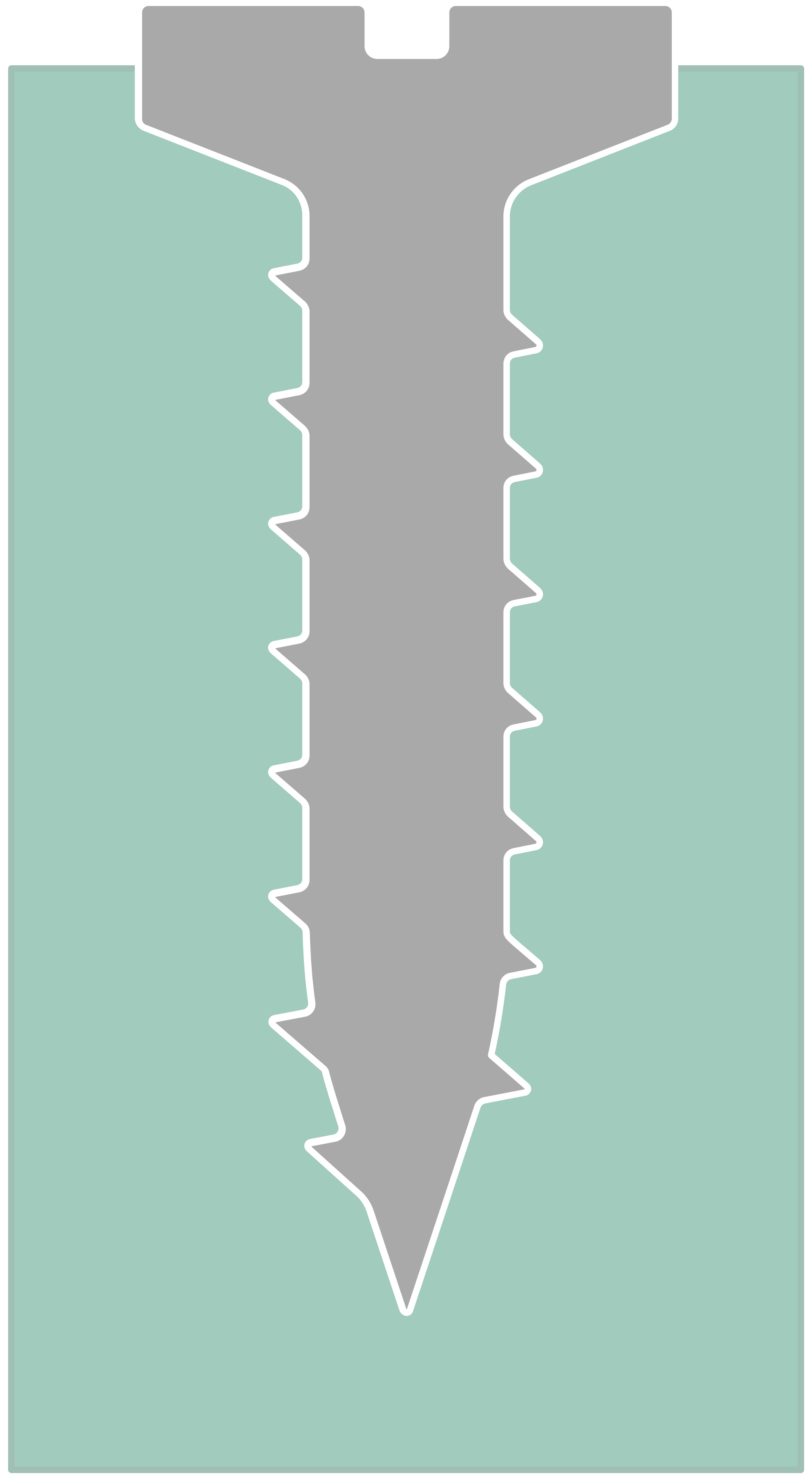


DIFFERENTIABILITY



PRIMAL TRADEOFFS

- Advantages
  - When # dofs < # constraints
  - Mass ratio insensitive
  - Easy to handle arbitrary nonlinear models
  - Good contact force distributions
  - Differentiable
- Disadvantages
  - Need to pick stiffness
  - Stiffness ratio sensitivity



FUTURE WORK

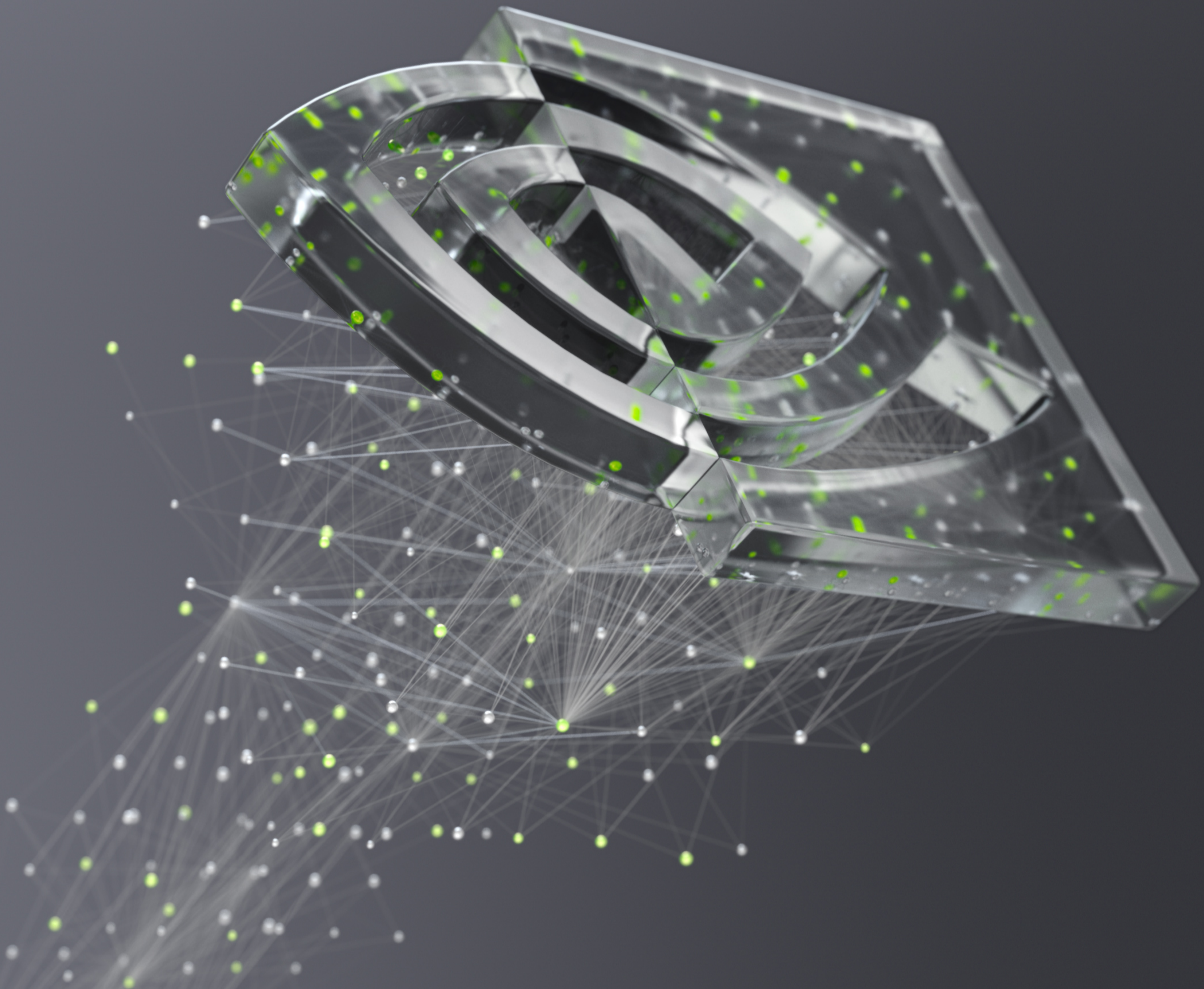
- Accelerated methods (Nesterov, Chebyshev, Anderson, nonlinear CG)
- Prefactorized preconditioners
- How important is smoothness on optimization?

SUMMARY

- Presented unified view of Projective and Position-Based Dynamics
- Sensitivity analysis
- Extended PD to rigid body contact
- Benchmarked on rigid body contact problems

REFERENCES

- *Projective Dynamics: Fusing Constraint Projections for Fast Simulation [Bouaziz 2014]*
- *Descent Methods for Elastic Body Simulation on the GPU [Wang et al. 2016]*
- *Fast and Stable Animation of Cloth with an Approximated Implicit Method [Kang et al. 2000]*
- *Interactive Animation of Structured Deformable Objects [Desbrun 1999]*
- *Continuous Penalty Forces [Tang 2012]*
- *Implicit Multibody Penalty-based Distributed Contact [Xu 2014]*
- *Convex and Analytically-Invertible Dynamics with Contact [Todorov 2014]*



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